Analysis of a New Bit Tracking Loop—SCCL

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Abstract—We propose a new bit tracking loop for biphase signals which is implemented like the MAP optimal bit synchronizer by the sample-correlate-choose-largest algorithm except that the estimator is sampled and moved at most one sample each bit time. A mathematical Markovian model for analysis is used. The performance of the bit tracking loop, the mean square error of the jitter and the average acquisition time, are theoretically derived. The numerical results of performance analysis for various signal-to-noise ratios are found through computer evaluations. The data obtained illustrate that this new structure is a very effective bit synchronizer for digital communications systems applying digital signal processing techniques.

I. INTRODUCTION

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it synchronization is an essential part of digital communications systems. It is necessary for the receiver to know the epoch of the incoming bit string so that it is able to make good decisions. The optimal bit synchronizer based on the MAP (maximum a posteriori probability) decision criterion has the structure shown in Fig. 1 for binary signals. If there are \(N\) channels in the optimal bit synchronizer (where, ideally \(N = \infty\)), this means that there are \(N\) hypotheses for the epoch, and \(N\) integrate-and-dump circuits are necessary. Implementation of this structure is impractical for many applications even with today’s VLSI technology. Several suboptimal bit synchronizers with structures which can be practically implemented have been proposed [1], [5], [6], [8], [10]–[12], [16]–[19], [21], [25], [26]. These structures are usually based on the following procedures. First, the received bit stream passes through a linear filter to reduce the noise effect and increase the observability of the bit transitions. Then the output of the linear filter is passed through a nonlinear filter. The direct way is to use even-law nonlinear filter(s) such as square type, absolute value type, or logcosh type, and combine delay or differentiating techniques to produce the spectral lines at the bit rate (and its harmonics). After low-pass filtering with a cutoff frequency equal to the bit rate, we are able to filter out the harmonics and extract the epoch information. There are two other well known structures—data transition tracking loop (DTTL) and early–late gate bit synchronizer, using two separate channels (in-phase and midphase channel for the DTTL; early gate and late gate for the early–late gate bit synchronizer) to produce spectral lines. Both structures are applied successfully in space communications. However, the DTTL suffers from the long transition time which makes the early–late gate bit synchronizer more popular for common digital communication systems. Further, joint synchronization and detection in specific channels has been investigated by Georgiades [27], [28]. Though Georgiades also demonstrated that estimating delay and sequence can yield unexpectedly good symbol error rate without a timing recovery unit [29], bit (or symbol) synchronization still provides advantages in performance and many aspects of communication system design.

With the advance of today’s microelectronic technology, more sophisticated bit (symbol) synchronizers become possible. This paper proposes a new structure based on a sample-correlate-choose-largest (SCCL) digital signal processing technique to more closely implement the optimal bit synchronizer. The goal is to obtain a better approximation to the optimal bit synchronizer than other structures. The block diagram and mathematical model of this new bit tracking

![Fig. 1. MAP optimal bit synchronizer.](image-url)
A first-order Markov chain is chosen as the mathematical model. The transition probabilities of the Markov chain model are derived in Section 11. The performance analysis of the new bit tracking loop with AWGN is given in Section IV, which also includes numerical performance results. The mean square error of the jitter and the average acquisition time of the bit tracking loop are presented for performance evaluation.

II. BIT TRACKING LOOP MODEL

From Fig. 1, we see that the optimal bit synchronizer for biphase signals is implemented as $N$ channels which represent $N$ hypotheses; the maximum output channel indicates the best epoch estimate. We select maximum every $M$ bits and reset in Fig. 1 [5]. If infinite observations are available, this optimal bit synchronizer based on MAP criterion can reach the best timing estimate (that is, estimate with the least time jitter). One disadvantage is the relatively complicated $N$-channel integrate-and-dump circuits. Our proposed approach is to decide the direction of the best estimator, and move the epoch estimate step by step to find the best estimate.

A. Three-Channel Model

Let us simplify the structure of Fig. 1 as shown by the suboptimal structure of Fig. 2. There are only three channels in this realization plus a unit to decide the bit timing by a sliding window structure. Although there are still $N$ hypotheses for bit timing, we consider only 3 at a time. We update the current timing (phase) among the previous bit timing and two immediate neighbor hypotheses of the previous decided timing for each bit period ($M = 1$). The function of the bit timing decision unit is to send timing signals to the three channels where the corresponding reference waveforms are generated. If the timing decision is $e_n$ for the previous bit, then the timing at the current bit is $e_n$ for the first channel; $e_0$ for the second channel; $e_+$ for the third channel.

A suboptimal digital realization with only one summation circuit equivalent to one integrate-and-dump circuit is shown as Fig. 3(a); the suboptimal structure makes a recursive bit timing decision based on one bit period. The logcosh function is not needed. It can be replaced by any even-law device or operation. We choose an absolute-value function here due to its simplicity in digital logic operation. The purpose of the analog filter is antialiasing of the noise. We assume biphase-level baseband signals for digital transmission as shown in Fig. 3(b).

The baseband waveform at the receiver $x(t)$ is composed of

$$x(t) = f(t) + n(t) \tag{2.1}$$

where $f(t)$ is a string of biphase signals which represent the information transmitted by the source, and $n(t)$ is additive white Gaussian noise with zero mean and one-sided spectral density $N_0$.

Suppose we use ideal bandpass filtering and sample $x(t)$ at the Nyquist rate so that the noise samples are independent. Let these samples be represented by $x_1, x_2, \ldots, x_t, \ldots$. 

Fig. 3 (a) Block diagram of SCCL (b) biphase-level signals.
for the movement of the estimator in the Markov chain is the direction of the shortest distance to state 1. It is possible that the estimator moves in the wrong direction due to noise and other factors discussed later. However, once in state 1, the estimator will stay in state 1 if there is no noise.

The mathematical model is concluded by the following statement. Let \( \tilde{t}_0(m) \) be the estimate at the \( m \)th bit period. Then, \( \{\tilde{t}_0(m)\}_{m=1}^{\infty} \) is a Markov process where

\[
\tilde{t}_0(m + 1) = \tilde{t}_0(m) + \arg \max_{i \in \{-1, 0, 1\}} |C_m(\tilde{t}_0 + i)|.
\]

Obviously, the sampled data can be used in the bit decision scheme. However, this topic is not within the scope of the paper.

### III. TRANSITION PROBABILITIES OF MARKOV CHAIN

The Markov model proposed in the previous section is used to analyze the behavior of the bit tracking loop. We first need to find the transition probabilities of the Markov chain to complete the description before proceeding with the analysis.

#### A. Mathematical Representation Of Transition Probabilities

We divide the incoming signal \( f(t) \) during two bit periods into four types based on two consecutive bits (the current bit and the next bit) as 11, 00, 10, and 01. We assume that the source transmits symbols 1 and 0 with equal probability. Further, we assume that the symbols in the symbol sequence are independent. Now we introduce a sample phase parameter \( \theta \), which ranges from \(-\frac{1}{2}t\) to \(\frac{1}{2}t\) of a sampling period (or of \(\frac{1}{2}t\) bit period) which is the fraction of the bit interval that the true bit transitions deviate from the sampling times. Obviously, \( \theta \) is a uniform random variable on that interval because we assume that the bit period is known exactly. \( \theta \), while random, is fixed for all time. The net effect of this \( \theta \) offset is an effective reduction in the energy per bit. For reasonably large value of \( N \), the effect can be neglected.

Let the normalized sampling times of each bit be \( N \) and \( N \) under the ideal synchronous situation. At the same time, the corresponding normalized timings (phases) of each bit at the local oscillator(s) are \( 0, \frac{1}{N}, \ldots, \frac{N}{N-1} \).

First, we compute \( q_{\theta}^{(n)}(x) \), the probability density function (pdf) of \( C_m(n) \). That is, the probability density function which corresponds to the situation while the current estimator falls in state \( n \) where the best estimator is in state 1. From (2.2) with \( \tilde{t}_0 = n \),

\[
C_m(n) = \sum_{k=1}^{N} g_k x_m N + k + n
\]

We can evaluate the performance of this bit tracking loop by analyzing the Markov chain. The best (or correct) direction
Type I: The combination of the consecutive bits is 11. Let the mean be
\[
\mu_1^n = \sum_{k=1}^{N} g_k f_{mN+k+n}^{11}
\]
\[
\mu_1^n = \begin{cases} 
[N - 4(n - 1)]A, & 1 \leq n < \frac{N}{2} + 1 \\
[-N + 4(n - \frac{N}{2} - 1)]A, & \frac{N}{2} + 1 \leq n \leq N
\end{cases}
\] (3.3)

Type II: The combination of the consecutive bits is 00. The result is just the negative value of Type I.
\[
\mu_2^n = \sum_{k=1}^{N} g_k f_{mN+k+n}^{00}
\]
\[
\mu_2^n = \begin{cases} 
[-N + 4(n - 1)]A, & 1 \leq n \leq \frac{N}{2} + 1 \\
[N - 4(n - \frac{N}{2} - 1)]A, & \frac{N}{2} + 2 \leq n \leq N
\end{cases}
\] (3.4)

Type III: The combination of consecutive bits is 10.
\[
\mu_3^n = \sum_{k=1}^{N} g_k f_{mN+k+n}^{10}
\]
\[
\mu_3^n = [N - 2(n - 1)]A \text{ for } 1 \leq n \leq N
\] (3.5)

Type IV: The combination of consecutive bits is 01. The result is just the negative value of Type III.
\[
\mu_4^n = \sum_{k=1}^{N} g_k f_{mN+k+n}^{01} = -\mu_3^n \text{ for } 1 \leq n \leq N
\] (3.6)

where \(A\) is the amplitude of the received signal without noise.

From (3.3)–(3.6) and above statement, we have
\[
q_0^n(x) = q_0^{n+1}(x) \text{ for } n = 1, \ldots, N \text{ } x \in R
\] (3.8)
\[
q_0^{-1}(x) = q_0^{n-1}(x) \text{ for } n = 1, \ldots, N \text{ } x \in R
\] (3.9)

Hereafter, we take \(n\) modulo \(N\) on \(1, \ldots, N\) so that if \(n = N\), then \(n + 1\) is defined to be 1, and if \(n = 1\), then \(n - 1\) is defined to be \(N\), such that the notations are able to represent the real situation of the Markov chain.

Decisions in our bit tracking loop are based on the absolute values of \(C_m(t_0 + i)\). The pdf of \(|C_m(t_0 + i)|\) given \(t_0 = n\) is
\[
p_n^c(x) = \Pr\{|C_m(t_0 + i)| = x | t_0 = n\}
\]
\[
= \begin{cases} 
q_n^c(x) + q_n^c(-x) & x \in R^+ \\
\mathbf{0} & \text{elsewhere}
\end{cases}
\] (3.10)

where \(R^+ = \{x | x \geq 0\}\).

Now, we compute the transition probabilities of the Markov chain. Since the estimate moves in the direction which makes the absolute value of \(C_m(t_0 + i)\) largest, we have the following formula for the transition probabilities.
\[
p_{n,n+1} = \Pr\{|C_m(t_0 + i)| \text{ is largest } | t_0 = n \text{ and } t_0 = 1\}
\]
\[
\text{for } i = -1, 0, 1 \text{ } n = 1, \ldots, N
\] (3.11)

where \(t_0 = 1\) means that the best estimator is in state 1.

At any given time this is equivalent to a ternary decision problem. There is a signal that we detect at three different points on the time axis, and (3.11) is the probability we decide to it be \(-1, 0, 1\) or, 0, or equivalently, the estimator moves backward, forward, or stays in the same state. Then, we intend to decide which hypothesis \(H_i\) is true.

\(H_i\): the correct synchronization is in the direction of \(i = -1, 0, 1\). Thus, we have
\[
p_{n,n} = \int dy_2 \int dy_1 \int dy_3 d_n(y_1, y_2, y_3)
\]
\[
\text{for } i = -1, 0, 1 \text{ } n = 1, \ldots, N
\] (3.12)

where \(d_n(y_1, y_2, y_3)\) is the joint pdf of \(C_m(t_0 + i), i = -1, 0, 1\).

B. Integration Regions

Before computing (3.12), we observe some symmetry properties of the integration regions in (3.12). The three-dimensional space is divided into 6 identical pyramids with a common vertex. If the axis of one pyramid is the \(y_3\) axis, the pyramid belongs to \(\Omega_i\). The plane vertical to the \(y_3\) axis intersects the pyramid and forms a rectangle.
Therefore,

\[
\begin{align*}
\int_{\Omega_1} &= \int_{-\infty}^{\infty} \int_{-\infty}^{y_1} \int_{-\infty}^{y_3} dy_1 dy_2 dy_3 \\
\int_{\Omega_2} &= \int_{-\infty}^{\infty} \int_{-\infty}^{y_2} \int_{-\infty}^{y_3} dy_2 dy_3 dy_1 \\
\int_{\Omega_3} &= \int_{-\infty}^{\infty} \int_{-\infty}^{y_3} \int_{-\infty}^{y_2} dy_3 dy_1 dy_2
\end{align*}
\]

(3.13)

where \( \Omega_i = \{ (y_1, y_2, y_3) : y_i \) has the largest absolute value among \( y_1, y_2, y_3 \).}

C. Joint PDF

The remaining work to complete the description of (3.12) is to find the joint pdf \( d_n(y_1, y_2, y_3) \). Suppose the current estimator is in state \( n \). Define random variables

\[
\begin{align*}
Y_1 &= C_m(n - 1) \\
Y_2 &= C_m(n) \\
Y_3 &= C_m(n + 1)
\end{align*}
\]

The marginal probability density functions of \( Y_i \)'s are (3.7). The correlation coefficients of \( Y_1, Y_2, Y_3 \) are shown in Appendix B to be

\[
\rho_{12} = \rho_{21} = \frac{N - 3}{N}, \quad \rho_{23} = \rho_{32} = \frac{N - 3}{N}, \quad \rho_{13} = \rho_{31} = \frac{N - 6}{N}
\]

and the covariance matrix \( \hat{Q} \) is

\[
\hat{Q} = NB \begin{pmatrix}
\frac{1}{N^2} & \frac{N - 3}{N^2} & \frac{N - 6}{N^2} \\
\frac{N - 3}{N^2} & \frac{1}{N^2} & \frac{N - 3}{N^2} \\
\frac{N - 6}{N^2} & \frac{N - 3}{N^2} & \frac{1}{N^2}
\end{pmatrix}
\]

(3.14)

Let the sample vector be

\[
\tilde{y}^n = (y_1 - \mu_1, y_2 - \mu_2, y_3 - \mu_3)
\]

The transition probabilities are given by integrating on the appropriate decision regions and summing on bit patterns.

\[
\begin{align*}
p_{n-1} &= \int_{\Omega_1} \left( \frac{1}{4} (2\pi)^{-3/2} |Q|^{-1/2} \sum_{j=1}^{4} e^{-1/2 \tilde{y}^n \cdot \hat{Q}^{-1} \tilde{y}^n} \right) d\tilde{y} \\
p_n &= \int_{\Omega_2} \left( \frac{1}{4} (2\pi)^{-3/2} |Q|^{-1/2} \sum_{j=1}^{4} e^{-1/2 \tilde{y}^n \cdot \hat{Q}^{-1} \tilde{y}^n} \right) d\tilde{y} \\
p_{n+1} &= \int_{\Omega_3} \left( \frac{1}{4} (2\pi)^{-3/2} |Q|^{-1/2} \sum_{j=1}^{4} e^{-1/2 \tilde{y}^n \cdot \hat{Q}^{-1} \tilde{y}^n} \right) d\tilde{y}
\end{align*}
\]

(3.15)

where \( |Q| \) is the determinant of the matrix \( Q \), and \( d\tilde{y} = dy_1 dy_2 dy_3 \).

Careful readers may notice that our formalism in not completely correct for \( n = 1 \) and \( n = N \). This weakness has small effect on our analysis. Corrections and discussions are given in Appendix C.

IV. PERFORMANCE ANALYSIS

In this section, the equations which represent the behavior of the bit tracking loop are formulated and solved.

A. Mean Square Error

Once we have the transition probabilities of the Markov chain, we are able to construct an \( N \) by \( N \) transition probability matrix \( \tilde{P} \) as

\[
\tilde{P} = \begin{pmatrix}
p_{1,1} & p_{1,2} & \cdots & p_{1,n} \\
p_{2,1} & p_{2,2} & \cdots & p_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
p_{n,1} & p_{n,2} & \cdots & p_{n,n}
\end{pmatrix}
\]

Let \( \tilde{\pi} = [\pi_1, \pi_2, \ldots, \pi_N] \) be the steady-state probability vector. Then, we can use (4.1) and (4.2) to solve the steady-state probabilities.

\[
\sum_{n=1}^{N} \pi_n = 1. \quad (4.1)
\]

Let \( t_o \) be the actual bit transition time. The mean square error with fixed \( \theta \) in steady state is

\[
\sigma^2 = E_{\theta}\{(t_0 - t_o)^2\} = \sum_{n=1}^{N} \pi_n n^2 + 4N - 2N + \frac{N^2}{4}
\]

(4.2)

If \( \theta \) is assumed to be uniformly distributed, then

\[
\sigma^2 = E\{(t_0 - t_o)^2\} = \int_{-\frac{1}{2}}^{\frac{1}{2}} \sigma^2_{\theta} d\theta = \sum_{n=1}^{N} \pi_n n^2 + 4N - 2N + \frac{N^2}{4}
\]

(4.3)

B. Mean Acquisition Time

Next, we compute the average number of bits to go to state 1. It is equivalent to the first passage time problem of Markov chain. Because this Markov chain is finite-state and
every state can commute with each other state, we conclude
[3] as follows.
1) This finite state Markov chain is aperiodic, irreducible,
recurrent, and uniquely closed.
2) This Markov chain is ergodic, and each state is persist-
ent.
3) A steady-state probability distribution exists.
We define the following parameters.
\( f_{m,1}^n \): the probability from state \( n \) into state 1 for the first
time through exactly \( m \) transitions
\[
f_{m,1}^n = \sum_{m=0}^{\infty} f_{m,1}^n \\
f_{0,1}^n = 1.
\]
Then the mean acquisition time from state \( n \) is
\[
F_n = \sum_m m f_{m,1}^n. \tag{4.5}
\]
From the transition matrix, we have that the overall mean
acquisition time is [15]
\[
F_n = p_{n,n-1}(F_{n-1} + 1) + p_{n,n}(F_{n} + 1) + p_{n,n+1}(F_{n+1} + 1)
\tag{4.6}
for \( n = 2, \ldots, N \) with \( F_1 = 0 \).
We are able to solve \( F_n \) for each \( n \) because there are \( (n-1) \)
unknowns and \( (n-1) \) independent equations. The explicit
solution for \( F_n \)'s are solved in Appendix D.
Suppose the starting synchronization time is uniformly
distributed over one bit period. In other words, the epoch falls
in each state with the same probability. So, the probability of
starting from state \( n \) is the same for each \( n \). Thus, the average
acquisition steps \( F \) is
\[
F = \frac{1}{N} \sum_{n=1}^{N} F_n. \tag{4.7}
\]

C. Limiting Performance
Consider the special case when signal-to-noise ratio is infi-
finite (without noise). The Markov chain shown in Fig. 5 under
this condition is the limiting case of Fig. 4. The transition
probabilities become (4.8) below.
We are able to analyze this situation exactly following
the same procedures. The values of \( F \) for different \( N \) are
computed numerically as shown in Table I. The steady state
mean square error of the jitter for ideal case is therefore \( \left( \frac{\pi}{N} \right)^2 \)
because the estimate definitely goes to and stays in state 1. We
get the limiting performance of the bit tracking loop. Notice
that there may be false lock on a half bit boundary due to
the consecutive bit pattern Type III and IV. This results in the
nonideal transition probabilities in (4.7) and \( F > \frac{N}{2} \). Fig. 6
illustrates this phenomena by showing the mean values of the
estimates in a continuous sense. The phase region, \((-90^\circ, 90^\circ)\),
is considered as the pull-in region of our proposed
loop.

D. Mean Slip Time
One important factor to judge a bit tracking loop is the
mean slip time, defined as the average number of bits for
the estimator out of the pull-in region once it is perfectly
synchronized in state 1. The derivation is the same as that
in [22, Section VI].

\[
\begin{align*}
p_{1,1} &= 1 \\
p_{n,n-1} &= 1 \\
p_{n,n+1} &= 1 \\
p_{n,n-1} &= 1/2 \\
p_{n,n+1} &= 1/2 \\
p_{n,n} &= 1/2 \\
p_{n,n-1} &= 1/2 \\
p_{n,n+1} &= 1/2 \\
p_{n,n} &= 1/2
\end{align*}
\tag{4.8}
\]
Mean Values

Mean Values Of Estimates

Fig. 6. Mean values of estimates; correlations.

E. Cycle Slipping Probability

If the timing (phase) estimate moves across the phase 180 (or -180) degree, there exists cycle slipping. The steady-state cycle slipping probability \( P_{cs} \) in the SCCL is given by:

\[
P_{cs} = P\left\{\theta_0(m + 1) = \frac{N}{2} + 1 | \theta_0(m) = \frac{N}{2} \right\} \cdot P\left\{\theta_0(m) = \frac{N}{2} \right\} \\
+ P\left\{\theta_0(m + 1) = \frac{N}{2} | \theta_0(m) = \frac{N}{2} + 1 \right\} \cdot P\left\{\theta_0(m) = \frac{N}{2} + 1 \right\}
\]

\[
= P_{\frac{\pi}{2}, \frac{\pi}{2} + \frac{\pi}{2}, \frac{\pi}{2} + 1, \frac{\pi}{2} + \frac{\pi}{2} + 1}.
\]

Different from the analog PLL's, SCCL will not suffer from cycle slipping to lose any bit in a symbol sequence if a shift register is used to store sampled data for bit decision. It is an advantage to apply baseband signal processing in communication systems.

F. Numerical Results

The computer evaluations of (4.3), (4.4), and (4.7) are shown in the Figs. 7 and 8 respectively. To confirm our analysis, computer simulations based on the same assumptions made in this paper are shown in the same figures. They illustrate that our analysis is able to predict the performance quite accurately. Most importantly, the analysis shows that this bit tracking loop has good tracking performance. Please notice that the root mean square error for \( \theta = 0 \) is much smaller than that for \( \theta = 0.5 \) when \( \frac{E_b}{N_0} \) is above 0 dB. This suggests that the self-noise [20] will dominate the performance of SCCL. In addition, we can use larger \( N \) to achieve smaller root mean square error at the price of longer mean acquisition time.

The mean slip time is depicted in the Fig. 9. The cycle slipping probability is depicted in the Fig. 10. The effect on the bit error rate of BPSK (binary phase shift keying) is shown in the Fig. 11. Because of the numerical difficulty from multiple integrations, some figures may not be accurate as indicated by small variations in some curves.

V. CONCLUSION

Because of the complexity of numerical computations, only the numerical result for \( N = 16 \) is presented. However, the results that we obtain illustrate that this new bit tracking loop, SCCL, has a good performance compared with other bit synchronizers [5], [9], [10], [14], [16]. The mean square jitter error is quite low. The average acquisition time is fairly good at the same time. Therefore, this bit tracking loop is worth further investigating.

Further investigations have been done very recently such as analysis for dependent noise samples [23] and generalization to NRZ (nonreturn-to-zero) signals [24]. Other interesting topics include self-noise, how to use the bit tracking loop associated with a specific modulation/demodulation method, the bit period \( T \) is not a known constant at the receiver, modifications for this structure, joint synchronization and detection based on this approach, and applications to spread spectrum communications and other potential fields.

This new bit tracking loop is suitable for VLSI implementation or digital signal processors in communication systems. With this encouraging analysis, a new solution for synchronization may be provided in this paper and succeeding research.

APPENDIX A

Proof: Obviously, it is possible to reach state \( n \) form state 1 without returning to state 1, and the probability of this event is \( \alpha \).
Once state \( n \) is reached, the probability of never returning to state \( i \) is \((1 - f_{n,i})\). Thus, the probability that from state 1 this bit tracking loop never returns to state 1 is at least \( \alpha(1 - f_{n,1}) \).

However, each state of the Markov chain is persistent. The probability of never returning is 0 for each state. Since \( \alpha > 0 \), \( f_{n,1} = 1 \).

Q.E.D.

**APPENDIX B**

Consider the random variables \( Z_1, Z_2, Z_3 \). Suppose \( i_0(m-1) = n \).

\[
Z_1 = \sum_{k=1}^{N} n_{(m-1)N+n+k-1} - \sum_{k=\frac{N}{2}+1}^{N} n_{(m-1)N+n+k-1}
\]

\[
Z_2 = \sum_{k=1}^{N} n_{(m-1)N+n+k} - \sum_{k=\frac{N}{2}+1}^{N} n_{(m-1)N+n+k}
\]

\[
Z_3 = \sum_{k=1}^{N} n_{(m-1)N+n+k+1} - \sum_{k=\frac{N}{2}+1}^{N} n_{(m-1)N+n+k+1}
\]

where \( n_i \)'s are i.i.d. Gaussian random variable with zero
mean and variance $N_0$ as the assumption of this paper. Their variances are

$$\text{Var}(Z_1) = \text{Var}(Z_2) = \text{Var}(Z_3) = NB N_0. \quad (B.2)$$

Further,

$$E[Z_1 Z_2] = \sum_{k=1}^{N-1} E\left\{n_2^2 (n_1 - 1) N + n + k\right\}$$

$$- E\left\{n_2^2 (n_1 - 1) N + n + k\right\}$$

$$+ \sum_{k=N}^{N} E\left\{n_2^2 (n_1 - 1) N + n + k\right\}$$

$$= (N - 3) BN_0 = E[Y_1 Y_2]. \quad (B.3)$$

By (C.2), (C.3), and the fact that $Y_i$'s and $Z_i$'s have the same variance,

$$\rho_{12} = \frac{E[Y_1 Y_2]}{\sqrt{\text{Var}(Y_1) \text{Var}(Y_2)}}. \quad (B.4)$$

All the other correlation coefficients can be computed by exactly the same method.

**APPENDIX C**

**EXACT CONSIDERATIONS**

The noise sequence $\{n_1\}$ is independent and identically distributed. Examining (3.1), it is seen that there is a slight overlap between the noise samples in the ternary decision correlation metrics for the $(m + 1)$st bit and the $m$th bit except when the sample time is advanced. These overlaps could be eliminated by shortening the summation interval by two samples; the effect however, is negligible. If the samples are retained (as we have done), the synchronization sequence is not quite first-order Markov; if the samples are deleted, the decision sequence is truly first order Markov but the effective $E_{\mu}$ is reduced by a factor of $\frac{N-2}{N}$. In either case, the net effect can be neglected.

There is similar end effect for correlation metrics at the extreme $n = 1$ and $n = N$. Here there are actually three bits involved. The complete description of joint pdf $d_n$ is stated as follows.

Considering that $\mu_j n^{-1}$ actually involves three bits when
\[ d_1(y_1, y_2, y_3) = \frac{1}{8} (2\pi)^{-3/2} Q^{-1/2} \]
\[ \cdot \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} e^{-1/2 g^{ij,k} \cdot Q^{-1} \cdot g^{ij,k}} \]  
\[ (C.1a) \]

\[ d_N(y_1, y_2, y_3) = \frac{1}{8} (2\pi)^{-3/2} Q^{-1/2} \]
\[ \cdot \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} e^{-1/2 g^{ij,k} \cdot Q^{-1} \cdot g^{ij,k}} \]  
\[ (C.1b) \]

where
\[ g^{0,0} = (y_1 - \mu_{kij}, y_2 - \mu_{ij}, y_3 - \mu_{ijk}) \]
\[ g^{0,N-1} = (y_1 - \mu_{kij}, y_2 - \mu_{ij}, y_3 - \mu_{ijk}) \]
\[ g^{N,0} = (y_1 - \mu_{kij}, y_2 - \mu_{ij}, y_3 - \mu_{ijk}) \]
\[ g^{N,1} = (y_1 - \mu_{kij}, y_2 - \mu_{ij}, y_3 - \mu_{ijk}) \]

and
\[ P^{0,0} = \mu_{kij}^2 \]
\[ P^{0,N-1} = \mu_{ij}^2 \]
\[ P^{N,0} = \mu_{ij}^2 \]
\[ P^{N,1} = \mu_{ijk}^2 \]

where \( \mu_{kij} = \mu_1, \mu_{ij} = \mu_2, \mu_{ij} = \mu_3, \mu_{ijk} = \mu_4 \) as (3.3)–(3.6). Notice the correspondence between \( i, j, k \) and the order of the coming bits. If we neglect the end effect which is small for large enough \( N \), the transition probabilities are represented by

\[ P^{n,n-1} = \Pr \{ i_0(m) = n | i_0(m - 1) = n \} \]
\[ = \int dy_1 \int dy_2 \int dy_3 d_n(y_1, y_2, y_3) \]
\[ \cdot \sum_{i=0}^{1} \sum_{j=0}^{1} \sum_{k=0}^{1} e^{-1/2 g^{ij,k} \cdot Q^{-1} \cdot g^{ij,k}} \]  
\[ (C.2) \]

**APPENDIX D**

**EXPLICIT SOLUTION OF (4.6)**

Equations (4.6) are actually \( N \)-unknown linear equations as follows.

\[ F_1 = 0 \]
\[ F_2 = 1 + p_{2,1} F_1 + p_{2,3} F_2 + p_{2,3} F_3 \]
\[ \vdots \]
\[ F_N = 1 + p_{N,N-1} F_{N-1} + p_{N,N} F_N + p_{N,1} F_1 \]
\[ (D.1) \]

Let
\[ z_i = F_i - F_{i-1} \]
\[ i = 2, \ldots, N \]

and
\[ z_2 = F_2. \]

Further, consider the symmetric feature of the Markov chain.

\[ F_i = F_{N-i+2} \]
\[ i = 2, \ldots, \frac{N}{2}. \]
\[ (D.2) \]

Equation (D.1) is reduced to a \( \frac{N}{2} + 1 \)-unknown linear equation as

\[ -1 = -p_{2,1} z_2 + p_{2,3} z_3 \]
\[ -1 = -p_{3,2} z_3 + p_{3,4} z_4 \]
\[ \vdots \]
\[ -1 = -p_{i-1,i} z_i + p_{i,i+1} z_{i+1} \]
\[ \vdots \]
\[ -1 = -p_{N-1,N} z_N + p_{N,N} z_{N-1} \]
\[ -1 = -p_{N/2+1,N/2} z_{N/2+1} \]
\[ \vdots \]
\[ -1 = -p_{N/2+1,N/2} z_{N/2+1} \]
\[ (D.3) \]

From (D.2), we have
\[ F_{N/2+2} = F_{N/2} \]
\[ z_{N/2+2} = F_{N/2+2} - F_{N/2+1} \]
\[ = F_{N/2} - F_{N/2+1} \]
\[ = -z_{N/2}. \]

Therefore, the last equation of (D.3) is replaced by

\[ 1 = p_{N/2+1,N/2+2} z_{N/2+2} + p_{N/2+1,N/2+1} \]
\[ \vdots \]
\[ 1 = p_{N/2+1,N/2+2} z_{N/2+2} + p_{N/2+1,N/2+1} \]
\[ \vdots \]
\[ 1 = p_{N/2+1,N/2+2} z_{N/2+2} + p_{N/2+1,N/2+1} \]
\[ (D.3) \]

Solving the last two equations of (D.3) gives us

\[ z_{N/2} = \frac{p_{N/2+1,N/2} + p_{N/2,N/2+1}}{p_{N/2+1,N/2} + p_{N/2,N/2-1} + p_{N/2+1,N/2+1} + p_{N/2+1,N/2-1}} = V \]
\[ z_{N/2+1} = \frac{p_{N/2+1,N/2} + p_{N/2,N/2-1} + p_{N/2+1,N/2+1} + p_{N/2+1,N/2-1}}{p_{N/2+1,N/2} + p_{N/2,N/2-1} + p_{N/2+1,N/2+1} + p_{N/2+1,N/2-1}} = W \]

Then, solve (D.3) recursively by

\[ z_i = \omega_i + \mu_i z_{i+1} \]
\[ i = \frac{N}{2} - 1, \ldots, 2 \]
\[ z_{N/2} = V \]

where

\[ \omega_i = \frac{1}{p_{i,i-1}} \]
\[ \mu_i = \frac{p_{i,i+1}}{p_{i,i-1}} \]

We have

\[ z_i = \sum_{j=i}^{N/2-1} \left( \prod_{k=j+1}^{N/2-1} \frac{\omega_k}{\mu_k} \right) \frac{1}{2} \sum_{j=i}^{N/2-1} \mu_j \]
\[ i = 2, \ldots, \frac{N}{2} - 1. \]
Now, we are ready to compute the $F_i$'s explicitly.

$$F_i = z_2 + \cdots + z_i = \sum_{i=2}^{N} \sum_{j=1}^{N/2-1} \left( \sum_{k=j+1}^{N/2-1} \mu_k \right) \omega_i + V \prod_{j=1}^{N/2-1} \mu_j$$

$$F_{N/2} = F_{N/2-1} + V$$

$$F_i = F_{i+2} \quad i = \frac{N}{2} + 2, \ldots, N$$

$$F_1 = 0.$$  \hspace{1cm} (D.4)

Mean slip time can be computed explicitly via the same approach.

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REFERENCES


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